The Structures of Zero-divisor Semigroups with Graph $K_n + 1$

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Abstract

In this paper, we determine the structures of zero-divisor semigroups whose graph is $K_n + 1$, the complete graph K_n together with an end vertex. We also present a formula to calculate the number of non-isomorphic zero-divisor semigroups corresponding to the complete graph K_n, for all positive integer n.

Key Words: Commutative zero-divisor semigroup, complete graph, complete graph with one end vertex

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1. Introduction

In this paper, we continue the work in [6] of studying semigroups determined by some graph G. In [6] it was proved that the complete graph K_n together

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with two end vertices has a unique corresponding zero-divisor semigroup, while the graph K_n together with more than two end vertices has no corresponding semigroups, for all $n \geq 4$. In [4] and [7], it was pointed out that both K_n and K_n together with one end vertex each has multiple corresponding zero-divisor semigroups. In this paper, we show a formula to calculate the number of non-isomorphic zero-divisor semigroups corresponding to the complete graph K_n, for all positive integer n. We determine the structure of zero-divisor semigroups whose graph is the complete graph K_n together with an end vertex. In fact, we have our discussions according to the four possible values of square of the end vertex x_1. In three cases, we obtain a simple formula for counting the number of mutually non-isomorphic zero-divisor semigroups corresponding to the complete graph $K_n + 1$. In the fourth case (i.e., $x_1^2 = x_1$), we give a simple necessary and sufficient condition and, we give a procedure for listing all the non-isomorphic zero-divisor semigroups corresponding to the complete graph $K_n + 1$.

For any semigroup S, following [3, 2, 5], associate to S a simple connected graph $\Gamma(S)$ whose vertex set is $T - \{0\}$, where $T = Z(S)$ is the set of all zero-divisors of S, with $x \neq y$ connected by an edge if $xy = 0$. Notice that T is an ideal of S and in particular, it is also a semigroup with the property that it consists of all zero-divisors of the semigroup T. We call such semigroups T *zero-divisor semigroups*. Obviously we have $\Gamma(S) \cong \Gamma(T)$. For a given connected simple graph G, if there exists a zero-divisor semigroup S such that $\Gamma(S) \cong G$, then we say that G *has corresponding semigroups*, and we call S a *semigroup determined by the graph* G. Other kinds of zero-divisor structure were studied in [7, 1]

All semigroups in this paper are multiplicative commutative zero-divisor semigroups with zero element 0, where $0x = 0$ for all $x \in S$, and all graphs in this paper are undirected simple and connected. Throughout this paper, we assume $n \geq 3$.

2. The graph K_n

Theorem 2.1. *For any n, denote $M_n = \{0, a_1, a_2, \cdots, a_n\}$. Then M_n is a zero-divisor semigroup corresponding to the complete graph K_n, if and only if M_n satisfies the following two conditions:*

(1) $a_i a_j = 0, \forall\ 1 \leq i \neq j \leq n$;

(2) $a_i^2 = 0$ *or* $a_i^2 = a_i$, *or* $a_i^2 = a_j$ *for some* $j \neq i$. *In the third case,* $a_j^2 = 0$.

Proof. We need only to prove the sufficiency part. By (1), we only need to check

the associative law, namely:

$$(a_ia_j)a_k = a_i(a_ja_k), \quad \forall 1 \le i, j, k \le n. \quad (*)$$

Case 1. If a_i, a_j, a_k are all the same or they are pairwise different, the associative law obviously holds.
Case 2. If $i = j, j \neq k$,the right hand of $(*)$ is 0 ,and the left of $(*)$ is $a_i^2a_k$, if $a_i^2 \neq a_k$, the left is 0 too; if $a_i^2 = a_k$, from (2), we can obtain $a_k^2 = 0$. Thus in this case the equality $(*)$ holds.
Case 3. If $i \neq j, i = k$, then the left side of $(*)$ is 0, and the right side is 0 too.
Case 4. If $i \neq j, j = k$, then the left hand of $(*)$ is 0, while the right side of $(*)$ is $a_ia_j^2$. If further $a_j^2 \neq a_i$, then the right side is 0. If $a_j^2 = a_i$, from (2) we again know $a_i^2 = 0$, so the right hand is 0 too. This completes the proof of Theorem 2.1. □

In the following Theorem 2.2, we denote by $p(j, i)$ the number of the following partitions of the integer j:

$$d_1 + d_2 + \cdots + d_i = j,$$

where $1 \le d_1 \le d_2 \le \cdots \le d_i$.

Theorem 2.2. *The number of non-isomorphic zero-divisor semigroups corresponding to the complete graph K_n is* $\sum_{k=1}^{n}\sum_{t=0}^{n-k} p(n-t, k) + 1$.

Proof. Since K_n is a complete graph, we have $a_ia_j = 0$ for all $i \neq j$, so we only need to decide the value of a_i^2. From Theorem 2.1 we can decompose the set $\{a_i \,|\, 1 \le i \le n\}$ into a union of the following three pairwise disjoint subsets:
(1) $A = \{a_i : a_i^2 = 0\}$;
(2) $B = \{a_i : a_i^2 = a_i\}$;
(3) $C = \{a_i : a_i^2 = a_j, a_j \in A\}$.

We assume that the cardinality of A (respectively, B) is $|A| = k\,(0 \le k \le n)$ (respectively, $|B| = t\,(0 \le t \le n-k)$). Without loss of generality, we assume the elements in A is $a_1, a_2, \cdots, a_k$. From Theorem 2.1 we know that $r^2 \in A$, $\forall\, r \in C$, so we can obtain if $k = 0$, then $t = n$. We assume there are λ_i elements $r \in C$ such that $r^2 = a_i (1 \le i \le k)$, then we get an equation:

$$\lambda_1 + \lambda_2 + \cdots + \lambda_k = n - t - k.$$

Let $\mu_1, \cdots, \mu_k$ be a permutation of $\lambda_1, \cdots, \lambda_k$ which satisfies $\mu_1 \le \mu_2 \le \cdots \le \mu_k$. For two zero-divisor semigroups S_1, S_2 whose zero-divisor graphs are K_n, it is

not difficult to see that S_1 is isomorphic to S_2 *if* and *only if* they have the same cardinalities $|A|, |B|, |C|$,and the same permutation $\mu_1, \cdots, \mu_k$. Thus in the following we assume that $0 \leq \lambda_1 \leq \cdots \leq \lambda_k$. So the number of solutions of the equation

$$\lambda_1 + \lambda_2 + \cdots + \lambda_k = n - t - k$$

where $0 \leq \lambda_1 \leq \cdots \leq \lambda_k$, is the number of corresponding isomorphic zero-divisor semigroups in case $|A| = k, |B| = t$. Now we substitute λ_i by $d_i - 1$ $(1 \leq i \leq k)$, then the above equation is equivalent to :

$$d_1 + \cdots + d_k = n - t \qquad (**)$$

where $1 \leq d_1 \leq \cdots \leq d_k, 0 \leq t \leq n - k$.

Finally, we denote by $p(n-t, k)$ the number of solutions of the equation above, then the number of zero-divisor semigroups corresponding to the complete graph K_n is $\sum_{k=1}^{n} \sum_{t=0}^{n-k} p(n-t, k) + 1$. $\square$

For any $n \geq 1$, denote

$$s(n) = \sum_{k=1}^{n} \sum_{t=0}^{n-k} p(n-t, k) + 1.$$

One can apply Theorems 2.1 and 2.2 to list all of the twelve (seven) non-isomorphic zero-divisor semigroups corresponding to K_4 (K_3, respectively). Thus $s(3) = 7, s(4) = 12$. These results will be applied in the last part of the next section.

3. The graph K_n with one end vertex

Throughout this section, let

$$M_n = \{0, a_1, \cdots, a_n\}, \quad M_{n,1} = \{0, x_1, a_1, \cdots, a_n\}.$$

and we assume that M_n is a semigroup with $\Gamma(M_n) \cong K_n$. If $M_{n,1}$ is a commutative zero-divisor semigroup with $\Gamma(M_{n,1}) \cong K_n + 1$, *the complete graph* K_n *together with an end vertex*, then we always assume $a_1 x_1 = 0$ and x_1 is an end vertex. In this case, M_n is an ideal of the semigroup $M_{n,1}$ by Theorem 4 of [4]. Thus we have the following necessary requirements for $M_{n,1}$:

(1) For any $2 \leq i \leq n$, $a_i^2 = 0$, or $a_i^2 = a_i$, or for some $j \neq i$, $a_i^2 = a_j$. In the case of $a_i^2 = a_j$, we also have $a_j^2 = 0$.

(2) $a_1^2 = a_1$ or $a_1^2 = 0$.

(3) $a_1x_1 = 0$, $a_ix_1 \in M_n - \{0\}, \forall i \neq 1$.

(4) $x_1^2 = 0, x_1$, or $x_1^2 = a_i, i = 1, 2, \cdots, n$. By symmetry, one need only consider the four cases of $x_1^2 = 0, x_1, a_1, a_2$.

In this section, we completely determine the structure of $M_{n,1}$ whose zero-divisor graph is $K_n + 1$, the complete graph K_n together with one end vertex. We have our discussions according to the possible value of x_1^2. We remark that for distinct values of x_1^2, the corresponding semigroups are not isomorphic.

Theorem 3.1. *Suppose in $M_{n,1}$ there is a multiplication such that $a_1x_1 = 0, a_ia_j = 0, \forall i \neq j$. Assume further $x_1^2 = 0$. Then $M_{n,1}$ is a semigroup whose zero-divisor graph is K_n together with an end vertex, if and only if the following conditions hold:*

(1) $a_1^2 = 0$.

(2) *For all $i \geq 2$, $a_ix_1 = a_1$.*

(3) *For all $i \geq 2$, $a_i^2 = 0$ or $a_i^2 = a_1$.*

In this situation, there are totally n mutually non-isomorphic commutative semigroups corresponding to the graph $K_n + 1$.

Proof. $\Longrightarrow$. Suppose $x_1^2 = 0, a_1x_1 = 0$ and assume $M_{n,1}$ is a commutative semigroup such that $\Gamma(M_{n,1}) \cong K_n + 1$. For any $i \geq 2$. since $(a_ix_1)x_1 = a_i(x_1^2) = 0$, we have $a_ix_1 \in \{a_1, x_1\} \cap M_n$. Thus $a_ix_1 = a_1$, $a_i^2x_1 = 0$. Hence $a_1^2 = 0$, and $a_i^2 \in \{0, a_1\}$.

$\Longleftarrow$. Consider the following associative law:

$$(uv)w = u(vw), \qquad \forall\ u, v, w \in M_{n,1} - \{0\} \qquad (*)$$

If x_1 does not occur in u, v, w, then (*) holds by Theorem 2.1. If $u = v = w = x_1$, then (*) also obviously holds. If exactly two of u, v, w are the x_1, then we have $(x_1v)x_1 = x_1(vx_1)$, and $0 = x_1^2w = x_1(x_1w)$ for $w \in M_n$ since either $x_1a_1 = 0$ or $x_1a_i = a_1$. In the following we assume that exactly one of the u, v, w is x_1.

Case 1. Assume $u = x_1$ (or equivalently, $w = x_1$). In this case, $0 = (x_1a_1)w$, while $x_1(a_1w) = x_10 = 0$. $(x_1a_j)a_i = a_1a_i = x_1(a_ja_i)$ holds for all $i \neq j$. For $i = j$, we have $(x_1a_i)a_i = 0 = x_1(a_i^2)$. Thus the multiplication is associative in this case.

Case 2. Assume $v = x_1$. If $u = a_1$, then $(a_1x_1)a_k = 0 = a_1(x_1a_k)$. If $u = a_2$, then $(a_2x_1)a_k = a_1a_k = 0 = a_2(x_1a_k)$.

The above discussions show that $M_{n,1}$ is a commutative zero-divisor semigroup, and $\Gamma(M_{n,1}) \cong K_n + 1$ if the conditions (1) to (3) hold. Finally, all the mutually non-isomorphic commutative zero-divisor semigroups corresponding to

K_n+1 are listed in the following

$$M_{n,1}^i = \{0, x_1, a_1, \cdots, a_n\}, \quad i = 1, 2, \cdots, n,$$

where $a_1x_1 = 0, x_1^2 = 0, a_ra_s = 0, \forall 1 \le r \ne s \le n$, $a_kx_1 = a_1, \forall 2 \le k \le n$, and $a_j^2 = a_1, \forall 1 \le j \le i$ while $a_j^2 = 0, \forall i < j \le n$.

This completes the proof. $\square$

Theorem 3.2. *Suppose in $M_{n,1}$ there is a multiplication such that $a_1x_1 = 0, a_ia_j = 0, \forall i \ne j$. Assume further $x_1^2 = x_1$. Then $M_{n,1}$ is a semigroup whose zero-divisor graph is K_n+1, if and only if the following conditions hold:*

(1) *For all $i \ge 2$, $a_ix_1 \in M_n - \{a_1, 0\}$ and, there exists at least one $i \ge 2$ such that $a_ix_1 = a_i$.*

(2) If $a_j = a_ix_1$ $(2 \le j \ne i \le n)$, then $a_jx_1 = a_j$, $a_j^2 = 0$, and $a_i^2 = 0$ or $a_i^2 = a_1$;

(3) *If $a_rx_1 = a_r$ $(r \ge 2)$, then a_r^2 is equal to one of the following: $0, a_r, a_j$, where $2 \le j \ne r \le n$. If $a_rx_1 = a_r$ and $a_r^2 = a_j$ for some $2 \le j \ne r \le n$, then $a_jx_1 = a_j, a_j^2 = 0$.*

(4) *$a_1^2 = 0$ or $a_1^2 = a_1$. If $a_i^2 = a_1$ for some $i \ge 2$, then $a_1^2 = 0$.*

Proof. $\Longrightarrow$. Suppose $M_{n,1}$ is a commutative zero-divisor semigroup such that $\Gamma(M_{n,1}) \cong K_n+1$. Since $n \ge 3$, x_1 is an end vertex. Then from $0 = (x_1a_1)a_1 = x_1(a_1^2)$ we obtain (4), i.e., either $a_1^2 = a_1$ or $a_1^2 = 0$.

For any $i \ge 2$, we have $a_ix_1 \ne a_1$, by the assumption $x_1^2 = x_1$. Thus $a_ix_1 \in M_n - \{a_1, 0\}$ since M_n is an ideal of $M_{n,1}$. This proves the first part of (1). The second statement of (1) follows from (2). (2) follows easily from the conditions given.

If $a_rx_1 = a_r$ and $a_r^2 \ne 0, a_r$, then $r \ge 2$ and $a_r^2 = a_j$, where $1 \le j \ne r \le n$. If $j = 1$, then we have $0 = a_1x_1 = a_r^2x_1 = a_r^2$, a contradiction. This proves (3).

$\Longleftarrow$. We only need to check the equality

$$(uv)w = u(vw) \quad (*)$$

for all $u, v, w \in M_{n,1}$.

Case 1. If x_1 does not occur in u, v, w, then (*) holds by Theorem 2.1. If $u = v = w = x_1$, then (*) also obviously holds.

Case 2. Assume that exactly two of u, v, w are the x_1. Then we have $(x_1v)x_1 = x_1(vx_1)$. The only other case to verify is $(x_1x_1)a_i = x_1(x_1a_i)$, i.e., $x_1a_i = x_1(x_1a_i)$: If $i = 1$, then both sides equal to 0. If $i \ge 2$ and $a_ix_1 = a_i$, then both sides equal to a_i. If $i \ge 2$ and $a_ix_1 = a_j$ for some $j \ne i$, then $a_ix_1 = a_j = a_jx_1 = x_1(x_1a_i)$.

Case 3. Now assume that exactly one of the u, v, w is x_1. We need only check in the following two situations.

Subcase 3.1. Consider $(x_1v)w = x_1(vw)$. If $v = a_1$, then both sides equal 0 since by condition (4), $a_1^2 = 0$ *or* a_1.

If $v = a_i$ $(i \geq 2)$ and $a_ix_1 = a_i$, then $(x_1a_i)a_k = a_ia_k = x_1(a_ia_k)$: If $i \neq k$, then each side is equal to 0. If $i = k$, then $a_i^2 = x_1(a_i^2)$ since a_i^2 is equal to one of the following $0, a_i, a_j$ $(j \neq i)$, by condition (3).

If $v = a_i$ $(i \geq 2)$ and $a_ix_1 = a_j$ for some $j \neq i$, then the left side is $(x_1a_i)a_k = a_ja_k$, while the right side is $x_1(a_ia_k)$. When $j = k$, Then each side is equal to 0. When $j \neq k$, again each side is equal to 0 since $a_i^2 = 0$ or a_1 under assumption $a_ix_1 = a_j$ $(i \neq j)$.

Subcase 3.2. Finally, let us consider

$$(a_ix_1)a_k = a_i(x_1a_k) \qquad (**)$$

It is easy to verify $(a_1x_1)a_k = a_1(x_1a_k)$ for all k. In the following we assume $i \geq 2$. If $a_ix_1 = a_i$, then the left side of $(**)$ is a_ia_k and the right side is $a_i(x_1a_k)$. If further $i = k$, then both sides are a_i^2. If $i \neq k$, then both sides are 0. Finally, we assume $a_ix_1 = a_j$ for some $j \neq i$. Then the left side of $(**)$ is a_ja_k and the right side is $a_i(x_1a_k)$. If $j \neq k$, then each side is equal to 0 since $i \geq 2$, and $a_ix_i \neq x_i$. If $j = k$, then the left side is $a_j^2 = 0$ and the right side is $a_i(x_1a_j) = a_ia_j = 0$. This completes the whole verification. □

Theorem 3.3. *Suppose in $M_{n,1}$ there is a multiplication such that $a_1x_1 = 0, a_ia_j = 0, \forall i \neq j$.*

(i) *If in addition $x_1^2 = a_1$, then $M_{n,1}$ is a semigroup whose zero-divisor graph is K_n together with an end vertex, if and only if $a_rx_1 = a_1, \forall r \geq 2$ and $a_i^2 = 0$ for all i. In this situation, there is exactly one zero-divisor semigroup S with graph $\Gamma(S) \cong K_n + 1$.*

(ii) *If in addition $x_1^2 = a_2$, then $M_{n,1}$ is a semigroup whose zero-divisor graph is K_n together with an end vertex, if and only if the following conditions hold:*

(1) *$a_1^2 = 0$ and for any $i \geq 3$, $a_ix_1 = a_1$ and, $a_i^2 = 0$ or $a_i^2 = a_1$.*

(2) *Exactly one of the following cases occurs*:

(A) $a_2x_1 = a_1$, and $a_2^2 = 0$;

(B) $a_2x_1 = a_2$, *and* $a_2^2 = a_2$;

(C) $a_2x_1 = a_r$ *for some* $3 \leq r \leq n$, *and* $a_2^2 = a_1, a_r^2 = 0$.

In the case of (ii), there are totally $3n - 4$ mutually non-isomorphic commutative semigroups corresponding to the graph $K_n + 1$.

Proof. (i) Assume that $M_{n,1}$ is a semigroup whose zero-divisor graph is K_n

together with an end vertex. If in addition $x_1^2 = a_1$, then $a_1^2 = 0$ and for all $i \geq 2$, $(a_i x_1)x_1 = 0$. Thus $a_i x_1 = a_1, a_i^2 = 0, \forall i \geq 2$. Conversely, it is routine to verify that the associative law holds.

(ii) $\Longrightarrow$. Suppose that $M_{n,1}$ is a semigroup whose zero-divisor graph is K_n together with an end vertex and assume $x_1^2 = a_2$ Then (1) follows easily from the assumption. Since M_n is an ideal of $M_{n,1}$, thus $a_2 x_1 \in M_n - \{0\}$. If $a_2 x_1 = a_1$, then $a_2^2 = (a_2 x_1)x_1 = a_1 x_1 = 0$. This proves (A). In a similar manner, one obtains (B) and (C).

$\Longleftarrow$. Again we need only check the equality

$$(uv)w = u(vw) \quad (*)$$

for all $u, v, w \in M_{n,1}$.

Case 1. If x_1 does not occur in u, v, w, then (*) holds by Theorem 2.1. If $u = v = w = x_1$, then (*) also obviously holds.

Case 2. Assume that exactly two of u, v, w are the x_1. Then we need only to verify $a_2 a_i = x_1(x_1 a_i)$ since $(x_1 x_1)a_i = a_2 a_i$. In fact, if $i = 1$, then both are 0. If $i \geq 3$, then both sides are 0. If $i = 2$, then we need to verify $a_i^2 = x_1(x_1 a_2)$. This is the case by the assumption of (A),or (B), or (C).

Case 3. Assume that exactly one of the u, v, w is x_1. Then we need to verify both $(x_1 a_i)a_j = x_1(a_i a_j)$ and $(a_i x_1)a_j = a_i(x_1 a_j)$. In the following we only verify the first equality because the verifications of the second one is similar.

Consider the possible equality $(x_1 a_i)a_j = x_1(a_i a_j)$:

(1) If $i = 1$, then each side is equal to 0 since $a_1^2 = 0$.

(2) If $i \geq 3$, then the left side is 0. If in addition, $i = j$, then the right side is $x_1 a_i^2 = 0$ since $a_i^2 = 0$ or $a_i^2 = a_1$. If $i \neq j$, then $a_i a_j = 0$.

(3) The last subcase is $i = 2$. If $a_2 x_1 = a_1$, then each side is equal to 0. If $a_2 x_1 = a_2$, then $(x_1 a_2)a_2 = a_2 a_2 = a_2 = x_1(a_2 a_2)$, and $(x_1 a_2)a_j = a_2 a_j = 0 = x_1(a_2 a_j)$ for $j \neq 2$. If $a_2 x_1 = a_r$ with $r \geq 3$, then $(x_1 a_2)a_2 = a_r a_2 = 0 = x_1 a_1 = x_1(a_2 a_2)$, and for $j \neq 2$, $(x_1 a_2)a_j = a_r a_j = 0 = x_1 0 = x_1(a_2 a_j)$.

Finally, cases (A) and (B) each has $n-1$ mutually non-isomorphic commutative semigroups corresponding to the graph $K_n + 1$. In case (C) we have mutually $n-2$ non-isomorphic corresponding commutative semigroups. This completes the proof. □

We end up this paper with the following remarks.

Remark 1. Denote by $k_2(n)$ the number of mutually non-isomorphic commutative semigroups corresponding to the graph $K_n + 1$ in Theorem 3.2. Then by

Theorems 3.1 to 3.3, $k_2(n)+4n-3$ *is the total number of mutually non-isomorphic commutative semigroups corresponding to the graph* K_n+1.

Remark 2. In the following we provide a procedure for calculating $k_2(n)$. In Theorem 3.2, let $k_2(n,r)$ be the number of mutually non-isomorphic commutative semigroups corresponding to the graph K_n+1, in which there are exactly r numbers $i\in\{2,\cdots,n\}$ such that $a_ix_1=a_i$, where $r=1,2,\cdots,n-1$. Then

$$k_2(n)=\sum_{r=1}^{n-1}k_2(n,r).$$

When $r=n-1$, we have $a_ix_1=a_i$ for all $2\le i\le n$. In this subcase, the value of a_1^2 is either 0 or a_1, the value of a_i^2 is 0, a_i ora_j $(2\le i\ne j\le n)$. By Theorem 3.2 and Theorem 2.1, $k_2(n,n-1)=2s(n-1)$, where $s(n-1)$ is the number of non-isomorphic zero-divisor semigroups corresponding to the complete graph K_{n-1}.

When $r=1$, we can assume $a_2x_1=a_2$. For all $i\ge 3$, we obtain $a_ix_1=a_2$ by condition (3) of Theorem 3.2 and hence, $a_i^2=0$ or $a_i^2=a_1$. Furthermore, $a_1^2=a_1$ only if $a_i^2=0$ holds for all $i\ge 3$. Finally, it is routine to check that there are n mutually non-isomorphic associative multiplication tables in $M_{n,1}$ such that $\Gamma(M_{n,1})\cong K_n+1$. Hence $k_2(n,1)=n$.

When $r=2$, without loss of generality we can assume $a_ix_1=a_i, i=2,3$. Then by condition (3) of Theorem 3.2, we have $a_jx_1=a_2$, or $a_jx_1=a_3$ for all $j\ge 4$. If $\{a_2,a_3\}=\{a_jx_1\,|\,4\le j\le n\}$, then $a_2^2=0, a_3^2=0$. For any $4\le i\le n$, either $a_i^2=0$ or $a_i^2=a_1$ by condition (2), and $a_1^2=a_1$ only if $a_i^2=0,\forall i\ge 4$. Thus in this case, we have $n-1$ multiplication tables. The only other case is $\{a_2\}=\{a_jx_1\,|\,4\le j\le n\}$. In this case if $n\ge 4$, we have $3(n-1)$ multiplication tables since the value of a_3^2 could be one of $0, a_2$ or a_3. When $r=2, n\ge 5$ and $x_1^2=x_1$, there are totally $4(n-1)$ mutually non-isomorphic associative multiplication tables in $M_{n,1}$ such that $\Gamma(M_{n,1})\cong K_n+1$. Hence

$$k_2(n,2)=\begin{cases}3, & \text{if } n=3\\ 3\times(4-1), & \text{if } n=4\\ 4(n-1), & \text{if } n\ge 5.\end{cases}$$

For $r=3,4,\cdots,n-2$, one can continue these discussions. When $r=3$, like the $r=2$ case, there are five results for $n=3,4,5,6,\ge 7$ respectively.

If n is small, it is not very difficult to calculate the number $k_2(n)$. For example, $k_2(3)=6$ and $k_2(4)=4+3\times(4-1)+2\times 7=27$, $k_2(5)=5+4\times(5-1)+2\times 7+2\times 12=59$. Thus K_3+1, K_4+1 and K_5+1 has $15, 40$ and 76 mutually

non-isomorphic commutative semigroups, respectively. For general n, we still do not know if there is a simple formula for calculating $k_2(n)$.

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